Nancy Mae Eagles
Evans 854
nm.eagles@berkeley.edu

The Conway Knot is Not Slice

[st3ms] Seminar, Spring 22

April 10, 2023

These are the notes that I used to give a talk in the student-led 3-manifold seminar at UC Berkeley, fall 2023, demonstrating that the Conway knot is not slice. The result was famously proved by Lisa Piccirillo in 2018, and appeared in the Annals in 2020. As is the nature of a chalk talk, these accompanying notes are very rough and may contain typos, approximations, and errors.

1 Preliminary Definitions

1.1 Concordance and Slice

Knot theory concerns itself with understanding complexity of a knot. There are tons of invariants that say something about this, like the classical polynomials: Jones, Alexander, Conway, HOMFLY-PT. There are more geometric measures of complexity, such as Seifert genus and slice genus. There are also the quantum invariants like knot floer homology, and Khovanov homology. Given a knot, the silly question to ask is how much work do I have to do to try and unknot this guy? The first result that one learns is that in dimension 3, there are some knots that cannot be unknotted. Knotting are detected by things like Seifert genus, polynomials, etc. Moving to one dimension higher though, the knotting disappears. There is a way to homotope the knot to change crossings so that eventually one gets to the unknot.

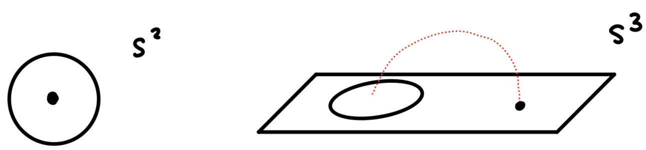


Figure 1: Unknotting analogue a dimension down.

In the picture above, the "crossing" is drawn a dimension down. Restricted to the plane, there is no way to unknot the two components, however adding a third dimension allows one to move the point out of the circle.
Though knotting is a codimension 2 phenomena, it has significant consequences on the topology of not just three manifolds, but four manifolds too. Descriptions of handlebodies arise as performing integral surgeries along knots living in the boundary of a three-sphere, and understanding the interactions between knots and tangles in this setting allows us to pictorially represent isomorphisms or more general maps between four manifolds. Conversely, the notions of topological vs smooth embeddings diverge in dimension 4, which allows us to explore subtle differences in pairs of knots by placing them in some ambient simply-connected 4-manifolds. Simply put, understanding 4-dimensional topology is the same as understanding knots.

Definition 1.1. Let $K_0, K_1 \subseteq S^3$ be two knots. Then K_1 is said to be *concordant* to K_2 if there exists a **smooth** embedding of the annulus $C : S^1 \times [0, 1] \hookrightarrow S^3 \times [0, 1]$ such that $C(S^1 \times \{0\}) = K_0$, and $C(S^1 \times \{1\}) = K_1$. We denote concordance by $K_0 \sim_c K_1$.

A knot is concordant to the unknot if and only if it bounds a disc in S^3. Hence, if $K \sim_c U$, then by capping the unknot off by a disc, we motivate a new definition:

Definition 1.2. Let $K_0 \subseteq S^3$ be a knot. Then K_0 is said to be *smoothly slice* if it bounds a disc in B^4, where we think of $S^3 = \partial B^4$. This is equivalent to K_0 being concordant to the unknot.

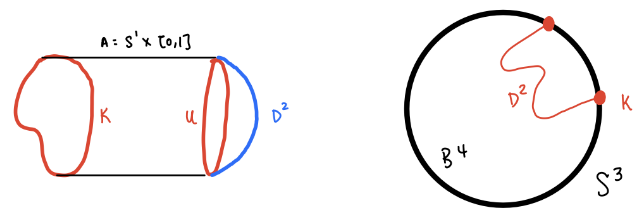


Figure 2: Concordance and sliceness.

Here notice that we said **smoothly** concordant or **smoothly** slice - there is an equivalent notion in the case that we only require **locally-flat** embeddings. We call this **topologically** concordant. If we were to ask simply for topological embeddings (homeomorphisms onto its image), then we would run into the problem that *every* knot would be topologically slice - any knot on the boundary S^3 bounds a topological disc in B^4, which we get by taking the cone on K_0 inside B^4.

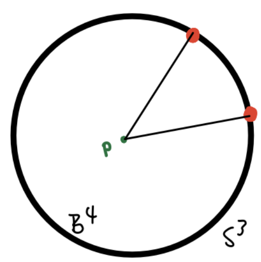


Figure 3: All knots are "topologically" slice if we're not careful!

One can alternatively state the above in terms of the *slice genus*:

Definition 1.3. The slice genus of a knot $K \hookrightarrow S^3$ is the minimal genus of a smoothly embedded surface $S \hookrightarrow B^4$ whose boundary is $\partial S = K$.

With this in mind, a knot is (smoothly) *slice* if and only it has slice genus 0.
Results from the 80s classify topologically slice knots:

Theorem 1.4 (Freedman 83)**.** Any knot with Alexander polynomial 1 is topologically slice.

The question is, does the work of Freedman extend to smooth sliceness? The answer is no - the result of Freedman (also due to work of Donaldson) is purely topological. In general, the answer is still no - we know that there are knots with Alexander polynomial 1 that are not smoothly slice. Showing this is very difficult, since many slice obstructions obstruct sliceness altogether.

1.2 Mutations of Knots

I'll turn our attention to a particular class of knots, namely mutations of knots.

Definition 1.5. Let K_0 be a knot in S^3. Suppose we can find a Conway sphere of K_0: that is, a smoothly embedded S^2 in S^3 which intersects K_0 in four distinct points. The Conway sphere separates S^3 into two 3-balls, $Int(S^2)$ and $Ext(S^2)$. Applying an involution to $Int(S^2)$ and gluing it back so that the tangles are connected, the new knot K_1 formed by this operation is called a *mutant* of K_0. If this new knot K_1 inherits a well-defined orientation from K_0, this knot is called a *positive mutant*.

The most classic examples of mutant knots are the Conway knot, and the Kinoshita-Teresaka knot. These are particularly of interest when discussing whether two knots are concordant. For

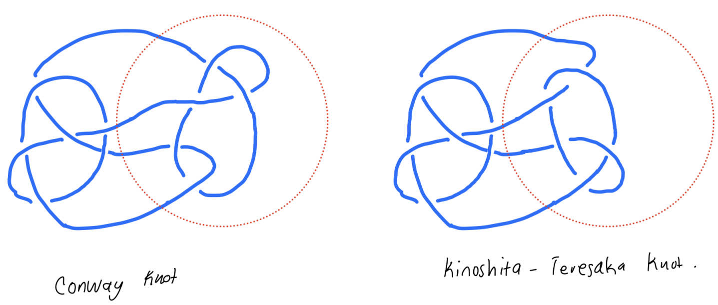


Figure 4: The Conway knot and the Kinoshita-Teresaka knot

example, we know that the Conway knot and Kinoshita-Teresaka knot have Seifert genus 3 and 2 respectively, but their concordance remained unknown for 50+ years. Results of (probably some guy but I can't find a reference oops) showed that the Kinoshita-Teresaka knot is slice (concordant to the unknot), but it was unknown whether the Conway knot is slice. Many knot invariants are preserved by mutations of knots, which makes tackling this problem very tricky. The genius of Piccirillo was to turn attention away from the Kinoshita-Teresaka knot, which we already knew to be slice, and to another knot.

2 The Conway knot is not slice

The fundamental theorem used in Piccirillo's proof is due to Rasmussen:

Theorem 2.1 (Rasmussen, 2010)**.** If K is slice, then $s(K) = 0$.

We will circle back to precisely what the s-invariant is (should we have time). Naively, one would want to tackle this problem by studying the mutant knot. But this is known to be slice, and none of the invariants that we have can distinguish sliceness between positive mutants (at least the s-invariant of a knot does not, in fact no abelian invariants can).
Instead, we can study an auxilliary knot closely related to the Conway knot, whose sliceness would be equivalent to sliceness of the Conway knot.

Piccirillo's approach is as follows:

1. Study 4-manifold equivalent definitions of sliceness. Show that a knot is slice if and only if its knot trace can be smoothly embedded in S^4.
2. Exploit this by finding a knot K' that
   a) has the same knot trace as the Conway knot C, and
   b) show that this knot is not slice.

By showing that this knot is not slice, since C has the same knot trace as K', its knot trace cannot be smoothly embedded in S^4, and hence the Conway knot is not slice either.

2.1 A 4-manifold repackaging of sliceness

We can repackage the notion of sliceness into one that concerns only smooth embeddings of 4-manifolds into other 4-manifolds.

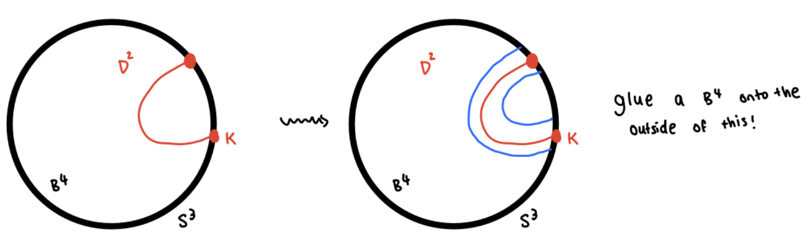
Definition 2.2. Let $K : S^1 \hookrightarrow S^3$ be a knot. Then the *trace* of a knot K is the 4-dimensional handlebody constructed by attaching a 0-framed 2-handle to a 4-ball.

We can construct the knot trace from a knot $K : S^1 \hookrightarrow S^3$ as follows. Consider the boundary of B^4 to be the copy of S^3, so that $K \hookrightarrow \partial B^4$. We can take a tubular neighbourhood of this knot in S^3, and let it be the attaching region of a 2-handle. More preceisely, we can think of attaching a 2-handle to a 4-ball in terms of the attaching circle, in this case K, along with a trivialisation of the normal bundle describing how one extends this attaching circle to an attaching region. One can think of the boundary of $X(K)$ as the 0-framed surgery on $K \hookrightarrow S^3$, $S^3_0(K) := (S^3 \setminus \nu_{K/S^3}) \cup (D^2 \times S^1)$.

Theorem 2.3. A knot is smoothly slice if and only if its knot trace $X(K)$ smoothly embeds into S^4.

This is a trickle down observation from the trace embedding lemma, originally given by Fox and Milnor in 1958. Luckily the proof of this is not too bad!

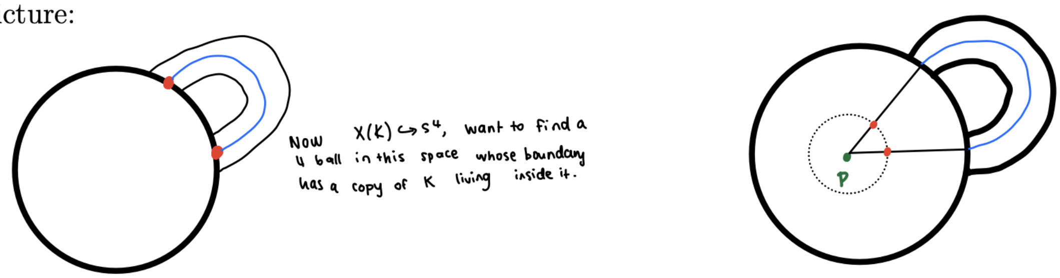
Proof. ($\implies$:) Suppose that K is slice. The schematic of this looks like



Then think of $S^4 = B_1 \cup_{S^3} B_2$, with $K : S^1 \hookrightarrow S^3$ living in this separating hyperspace. Since K is slice, K bounds a 2-disc in B_1, In particular, we can take a tubular neighbourhood of the disc D^2 bounded by K, which will give our handle. On the other side of S^3, we can glue B_2 to B_1, which gives us back S^4. Sitting inside all of this, we can see that we end up with a copy of the knot trace, given by B_2 along with this thickened disc bounded by K, and glued along K with

framing 0. You have to be a little careful about making sure that this actually has framing 0, but this simply just involves being a little bit careful about how you take the tubular neighbourhood of the bounding disc.
($\Longleftarrow$:) Let's suppose that the knot trace embeds in S^4. We want to show that K bounds a disc in B^4. A sort of naive idea would be to say that if we have an embedding $X(K) \hookrightarrow S^4$, then it is clearly not surjective, and so we can just restrict it to be some $X(K) \hookrightarrow B^4$. On the other hand, the core of the 2-handle on $X(K)$is a disc that has boundary K. So we may think to compose these maps and end up with some embedding of the disc whose boundary is K in B^4. But we run into some problems here with this argument, since a priori the smooth embedding of $X(K) \hookrightarrow S^4$ does not necessarily preserve as the boundary of the core as the knot K. In general, diffeomorphisms of spaces need not preserve knotting. We need to a little more careful about how we choose to restrict from S^4 to B^4.
Hence the way to circumnavigate this problem is to pass to a local picture, where one can make such a claim*. We know that we want to build a disc out of the core, since it seems the most natural, so we just have to choose a restriction of S^4 to B^4 that keeps the data of K. To do this then, look at the core of the handle, along with the cone on K in $X(K)$ sitting inside S^4. Since $X(K)$ smoothly embeds in S^4, then we can compose the inclusion of a piecewise-linear $S^2 \hookrightarrow S^4$, meaning that there is a copy of $S^2 \hookrightarrow S^4$, which fails to be smooth **only** at a singular point (the cone point p). Now look at $W := S^4 \setminus \nu(p)$. Then W is diffeomorphic to B^4, and K sits in the boundary of W. Moreover, the image of $S^2 \setminus \nu(p)$ is a copy of D^2. But this embedding remembers the structure of the cone, in particular, one can choose a neighbourhood $\nu(p)$ sufficiently small so that W intersects this copy of D^2 precisely in K.

Picture:



□

Alright, so this deals with the first part of Piccirillo's approach. Let's go on to the second part.

2.2 An auxiliary knot

Now that we've seen that a knot is slice if and only if its knot trace smoothly embeds into S^4, then one possible approach would be to study a knot which admits the same knot trace as the Conway knot, and study its sliceness. The first part of this is done using a more classical approach, i.e. RGB diagrams and kirby calculus, and the second is using Rasmussen's s-invariant for knots.

*I think to be precise about this statement, one would require some form of orientation preservation for the maps involved

2.2.1 RGB diagrams

Let L be the 3 component link in S^3

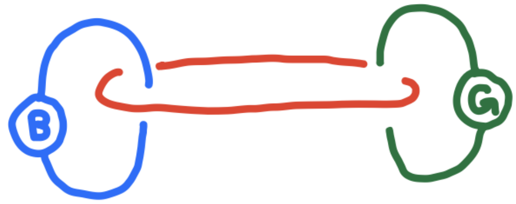


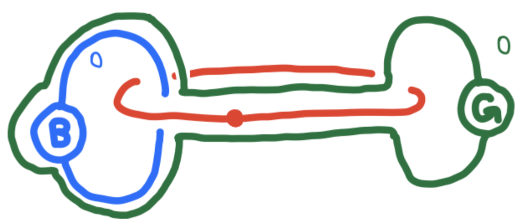
Figure 5: Link diagram of an RGB link.

Satisfying the following three properties:

1. L can be isotoped so that $B \cup R$ is isotopic to $B \cup \mu_B$,
2. L can be isotoped so that $G \cup R$ is isotopic to $G \cup \mu_G$, and
3. B and G have $lk(B, G) = 0$.

We can build a 4-manifold out of L as follows. Consider R as the attaching of a 1-handle in dotted circle notation, and B and G as the attaching circles of 0-framed 2-handles †
From L, one can construct two knots K and K' via handleslides, and what will arise will be attaching a 2-handle to a 4-ball with framing 0 (the knot trace).
The construction runs as follows:

1. Do a handleslide of G over B. Now R bounds a disc, D_R, with B intersecting D_R once, and G does not intersect D_R at all.



2. Cancel 0 surgery on B by the 1-handle R.

†If you haven't seen dotted circle notation before, well now you have! The idea though is that if you think about the attaching of a 1-handle to a 4-ball, the attaching spheres are actually two two spheres in S^3. But there's a way to cancel 1-handles by 2-handles, and so its actually enough to specify the attaching region of a 1-handle by the attaching region of the 2-handle that cancels it. We can treat R then as a sort of 0-framed 2-handle.

We can do the same thing while interchanging the roles of G and B and get another knot. Hence, this construction gives rise to two potentially distinct knots, whose knot traces are diffeomorphic:

Theorem 2.4. Via RGB-diagrams, $X(K) \simeq X \simeq X(K')$.

We can use this to create a new knot K' that shares a knot trace with the Conway knot C. From the construction above, it is not clear that we can do this for *all* knots - however it does work in the case that a knot has uncrossing number one.

Proposition 2.5. Let K be a knot with uncrossing number one. Then there exists a link L and a 4-manifold X such that $X \simeq X(K)$.

This is good for us, as the Conway knot has uncrossing number one.

Proof. Start with a blue diagram of K, which we will henceforth denote by B, with (wlog) positive uncrossing crossing c. Then let R be a pushoff of B with crossing c changed to negative, linking B with $-w(D)+2$ and a blackboard parallel of B outside of this neighbourhood of c. Let G be a meridian of R. We define X to be the 4-manifold with R representing a 1-handle in dotted circle notation, and B and G representing attaching circles of 0-framed 2-handles. Now, by choice G

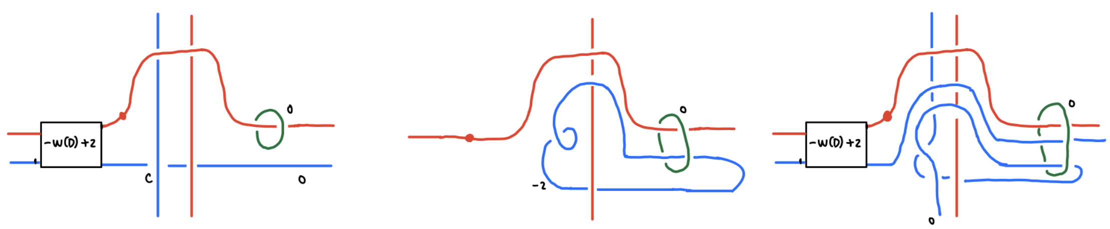


Figure 6: Construction of an RGB link for a knot with uncrossing number one.

and R are a cancelling pair, and so this 4-manifold is the same as the knot trace of B. What we want to do though is show that this actually arises from an *RGB* link (that we can get it into this form).
How do we go about that? Well, one can slide B over R to get the second frame. The knot diagrams of B and R have linking number -1, hence the resulting framing on B is -2. Here now, B is a meridian of R. We can introduce a Reidemeister I. move to B to get a little twist, and then do a handle slide of B over R again to get the third frame. One can then show that you can isotope R to be a meridian of the blue knot, and one can read off that $lk(B, G) = 0$. [‡]

□

With this result established, we can now apply it to the Conway knot. We will proceed as follows: take the Conway knot (which we will consider as the blue knot), run this construction from above, isotope it to a point where it exhibits the RGB link, perform the necessary handleslide and cancellation of the 1 and 2-handle pair, and arrive at another knot whose 0-framed surgery gives the same knot trace as the Conway knot.

[‡]A note on the Kirby calculus at play here: it took me a bit of time to really understand what was going on here, since it's hard to imagine this whole picture when we only have a small neighbourhood of a crossing in front of us. I recommend trying this construction with a simple knot such as the positive trefoil to see what's happening. There are also many different ways to find the linking number of two components, which can be confusing. The best way I thought about it was via orienting the knots in the canonical way, and the applying the algorithmic approach - see the wikipedia for linking number for this!

This auxiliary knot has diagram

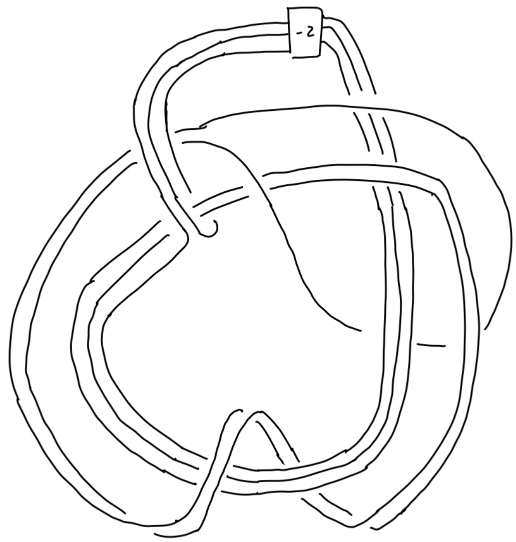


Figure 7: Auxiliary knot.

2.3 K' is not slice

Doing all this, we now have a a knot K' whose knot trace is diffeomorphic to that of the Conway knot. The final step in proving that this knot K' is not slice is done via Rasmussen's s-invariant of a knot, coming from Khovanov Homology. My goal for the rest of the talk is to give a very brief overview of what this invariant is, and what it does.
Recall the theorem from before:

Theorem 2.6 (Rasmussen, 2010)**.** If K is slice, then $s(K) = 0$.

More generally, the s-invariant gives a lower bound for the slice genus of a knot. I am unaware of the extend of how good this bound is, or when equality holds, but clearly in the case that it is slice, it must be that $s(K) = 0$, since slice genus is always nonnegative [§].
The general bound is: $|s(K)| \leq 2g_*(K)$.

2.3.1 A rough exposition of Khovanov Homology

Khovanov homology assigns to a knot K a homology group which arises from studying the cube of resolutions of a diagram representing K. To construct the cube of resolutions, take a knot (or link) K, and study all possible resolutions of the diagram. Resolutions form the vertices of the cube, and vertices are adjacent by edges relating two diagrams by a resolution change. If there are k crossings in K, then we can record the data of these resolved diagrams and their relations

[§]While reading Piccirillo's paper I became interested in maps between Khovanov homology for RGB- and mutant-related links. Due to Bar-Natan, Khovanov homology is invariant under positive mutation, and hence since the Kinoshita-Teresaka knot is slice, it has $s(K_{KT}) = 0$. Since K_{KT} and C are positive mutants, it follows that $s(C) = 0$. This disproves the converse of the theorem above.

via cobordisms by a k-dimensional cube. In particular, notice that two different resolutions of the same crossing are related by a saddle cobordism. A great example of this is drawn in Kauffman's *Introduction to Khovanov Homology*, Figure 2.

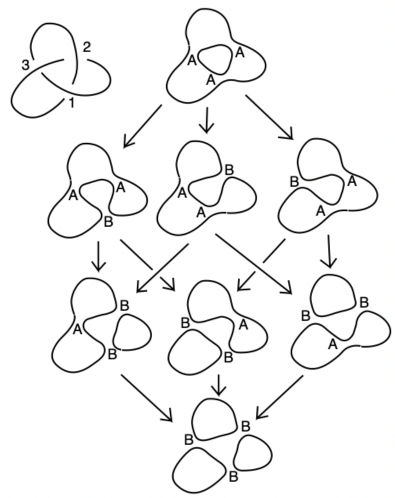


Figure 8: Cube of resolutions for the positive trefoil.

From this cube of resolutions, one can construct a chain complex (actually a cochain complex) by applying a 1+1 dimensional TQFT to the cube of resolutions. This just a fancy way of saying: replace each vertex of the cube by some group $A(v)$, and each edge by a map $A(e)$. Then $CKh(K) = \oplus_{v \in V} A(v)$, and for $x \in A(v)$,

$$d(v) = \bigoplus_{\{e \in E \ | \ v \in e\}} \pm A(e).$$

There is a homological grading of each element, given by

$$gr(x) = gr(v) = |v| - n_-,$$

where $|v|$ is the number of 1's in the coordinates of v, and n_- is the number of negative crossings in K ¶. In particular, d increases homological grading by 1. The resulting cohomology of this chain complex is defined to be the Khovanov Homology of K.

Using a slightly different TQFT, one can write a similar story and end up with a homology theory known as Lee's Homology of a knot, $LKh(K)$. These two homology theories are related by a spectral sequence. The grading on $CKh(K)$ carries through to Lee's complex, although it does not play quite so well with the boundary operator. However, it does define a filtration on the Lee complex, and unpacking this allows us to induce a grading on $LKh(K)$. Unpacking all of this, one can show that the minimum and maximum grading of a nontrivial element of $LKh(K)$ differs by 2, and hence we can define the s-invariant of K to be the number sitting in-between these gradings ‖.

¶Khovanov homology is really a bigraded homology theory, with the second grading called the *quantum* grading. For introductory work, see Kauffman's *Introduction to Khovanov Homology*, or actually in this case Rasmussen's *Khovanov Homology and the slice genus* is a very good reference. The only interesting thing I will say is that the boundary map must preserve the quantum grading of the vertex.

‖These sort of invariants are rife in quantum and floer theory, and can be used to detect subtle behaviour which classical invariants seem to miss. Most importantly for us, the s-invariant can almost detect slice genus

2.3.2 Returning to Piccirillo

Returning to the original problem at hand, we want to show that K' has $s(K') \neq 0$, so that K' and C are not slice. To do this, we need the following result:

Theorem 2.7. For any knot K, the generators of Lee Homology are located in gradings $(i, j) = (0, s(K) \pm 1)$. If K is slice, then $s(K) = 0$.

To prove the rest of the result, it suffices to compute the Khovanov/Lee homology of the knot, and identify the gradings in homology is nontrivial, and in which elements have homological grading $i = 0$.
There are computationally fast algorithms in which this can be done due to Bar-Natan, and the result shows that K' has Khovanov homology supported in gradings $(0, 3)$, $(0, 1)$ and $(0, -1)$. Hence, by the theorem, $s(K') \in \{0, 2\}$.
One can show that $s(K') = 2$ by studying the bigrading of Lee homology and its interplay with the spectral sequence arriving from Khovanov homology, which forces the quantum grading to be 2. In other words, $s(K') \neq 0$, and thus the Conway knot is not slice.

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